The rank two lattice type vertex operator algebras V_L^+ and their automorphism groups

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Abstract. Let L be a positive definite even lattice and V_L^+ be the fixed points of the lattice VOA V_L associated to L under an automorphism of V_L lifting the -1 isometry of L. For any positive rank, the full automorphism group of V_L^+ is determined if L does not have vectors of norms 2 or 4. For any L of rank 2, a set of generators and the full automorphism group of V_L^+ are determined.

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Contents

1 Introduction

This article continues a program to study automorphism groups of vertex operator algebras. See references in the survey [G2] and the more recent articles [G1], [DG1], [DG2], [DGR] and [DN1].

Here we investigate the fixed point subVOA of a lattice type VOA with respect to a group of order 2 lifting the -1 map on a positive definite lattice. We can obtain a definitive answer for the automorphism group of this subVOA in two extreme cases. The first is where the lattice has no vectors of norms 2 or 4, and the second is where the lattice has rank 2.

We use the standard notation V_L for a lattice VOA, based on the positive definite even integral lattice, L. For a subgroup G of $Aut(L)$, V_L^G denotes the subVOA of points fixed by G. When G is a group of order 2 lifting -1_L, it is customary to write V_L^+ for the fixed points (though, strictly speaking, G is defined only up to conjugacy; see the discussion in [DGH] or [GH]).

The rank 2 case is a natural extension of work on the rank 1 case, where $Aut(V_L^G)$ was determined for all rank 1 lattices L and all choices of finite group $G \leq Aut(V_L)$. The styles of proofs are different. In the rank 1 case, there was heavy analysis of the representation theory of the principal Virasoro subVOA on the ambient VOA. In the rank 2 case, there is a lot of work on idempotents, solving nonlinear equations as well as work with several subVOAs associated to Virasoro elements. For rank 2, the case of nontrivial degree 1 part is harder to settle than in rank 1.

Our strategy follows this model. Let V be one of our V_L^+. We get information about $G := Aut(V)$ by its action on the finite dimensional algebra $A := (V_2, 1^{st})$. We take a subset S of A which is G-invariant and understand S well enough to limit the possibilities for G (usually, there are no automorphisms besides the ones naturally inherited from V_L). A natural choice for S is the set of idempotents or conformal vectors. Usually, S spans A, or at least generates A. In the main case of rank 2 lattice, we prove that $Aut(V)$ fixes a subalgebra of A which is the natural $M(1)_2^+$. The structure of V is controlled by $M(1)^+$, which is generated by $M(1)_2^+$ and its eigenspaces, so we eventually determine G.

For several results, we give more than one proof.

For the case of a lattice L without roots, the automorphism group of V_L^+ was studied in the recent article [S].

We thank Harm Derksen for help with computer algebra.

2 Background Definitions and Notations

Notation 2.1. Let L be an even integral lattice. For an integer m, define $L_m := \{x \in L | (x,x) = 2m\}$. Let $H := \mathbb{C} \otimes L$, the ambient complex vector space. For a subset S of L, define $rank(S)$ to be the rank of the sublattice spanned by S.

Definition 2.2. For a lattice, L, the group of automorphisms of the free abelian group L which preserves the bilinear form is called *the group of automorphisms, the isometry group, the group of units* or *the orthogonal group of* L. This group is denoted $Aut(L)$ or $O(L)$. We will use the notation $O(L)$ in this article, as well as the associated $SO(L)$ for the elements of determinant 1, $PO(L)$ for $O(L)/\{\pm 1\}$ and $PSO(L)$ for $SO(L)/SO(L) \cap \{\pm 1\}$.

Definition 2.3. For an even integral lattice, L, we let $\hat{L}$ be the 2-fold cover of L described in [FLM], [DGH], [GH]. We may write bars for the map $\hat{L} \to L$. The *the group of automorphisms, the isometry group, the group of units* or *orthogonal group* is the set of group automorphisms of $\hat{L}$ which preserve the bilinear form on the quotient of $\hat{L}$ by the normal subgroup of order 2. It is denoted $Aut(\hat{L})$ or $O(\hat{L})$ and has shape $2^{rank(L)}.O(L)$. We use bars to denote the natural map $O(\hat{L}) \to O(L)$.

We list some notations for work with lattice type VOAs.

List of Notations

$\mathcal{D}(L)$	the discriminant group of the integral lattice L is $\mathcal{D}(L) := L^*/L$.
e^α	standard basis element for $\mathbb{C}[L]$
FVOA	framed vertex operator algebra [DGH]
LVOA	lattice vertex operator algebra [FLM]
LVOA type	the fixed points of a lattice vertex operator algebra under a finite group of automorphisms [DG1, DGR]
LVOA+	V_L^+ for an even lattice L
LVOAG(L)	the subgroup of $Aut(V_L)$, for an even integral lattice L, as described in [DN1]; it is denoted $\mathbb{N}(\hat{L})$ and is an extension of the form $T.Aut(L)$ (possibly nonsplit), where T is a natural copy of the torus $\mathbb{C} \otimes L/L^*$ obtained by exponentiating the maps $2\pi x_0$, for $x \in V_1$; the quotient of this group by the normal subgroup T is naturally isomorphic to $Aut(L)$. Also, $\mathbb{N}(\hat{L})$ is the product of subgroups TS, where $S \cong O(\hat{L})$ and $S \cap T = \{x \in T | x^2 = 1\} \cong \mathbb{Z}_2^{rank(L)}$. We may take S to be the

	centralizer in LVOAG(L) of a lift of -1; it has the form $2^{rank(L)}.Aut(L)$ and in fact any such S has this form. Denote the groups S, T by $\mathbb{O}(\hat{L})$ and $\mathbb{T}(\hat{L})$, respectively.
LVOA group for L	this means LVOAG(L).
LVOAG	this means LVOAG(L), for some L
LVOAG$^+(L)$	this is the centralizer in LVOAG(L) of a lift of -1 modulo the group of order 2 generated by the lift; it has the form $2^{rank(L)}.[Aut(L)/\langle -1\rangle]$; it is the *inherited group*
LVOAG$^+$	this means LVOAG$^+(L)$, for some L.
LVOA$^+$-group	same as LVOAG$^+$
$M(1), M(1)^+$	See Section 3.
$\mathbb{N}(\hat{L})$	See LVOAG(L)
o	linear map from V to $End(V)$
$\mathbb{O}(\hat{L})$	See LVOAG(L)
$\mathbb{T}(\hat{L})$	See LVOAG(L)
v_α	$e^\alpha + e^{-\alpha}$
$\mathbb{X}$ or $\mathbb{X}(L)$:	given an even integral lattice, L, this is a group of shape $2^{1+rank(L)}$ for which commutation corresponds to inner products modulo 2; see an appendix of [GH].
$\mathbb{XO}$ or $\mathbb{XO}(\hat{L})$	an extension of $\mathbb{X}$ upwards by $O(L)$.
$\mathbb{XPO}$ or $\mathbb{XPO}(\hat{L})$	a quotient of $\mathbb{XO}$ by a central involution which corresponds to -1_L under the natural epimorphism to $O(L)$.

Remark 2.4. If $(L, L) \subset 2\mathbb{Z}$, $\hat{L} \cong L \times \langle \pm 1\rangle$. Thus $O(\hat{L})$ contains a copy of $O(L)$ which complements the normal subgroup of order $2^{rank(L)}$ consisting of automorphisms which are trivial on the quotient group L of $\hat{L}$. This splitting passes to the groups $PO(\hat{L})$ and $\mathbb{XPO}(L)$.

3 Automorphism group of V_L^+ with $L_1 = L_2 = \emptyset$

In this section, we determine the automorphism group of V_L^+ with $L_1 = L_2 = \emptyset$ and assume only that $rank(L) > 1$. The automorphism group of V_L^+ in the case $rank(L) = 1$ is determined in [DG1] without any restriction on L. The assumption that $L_1 = L_2 = \emptyset$ ensures that any automorphism of V_L^+ preserves the subspace $M(1)_2^+$, which can be identified with the Jordan algebra S^2H.

Since $M(1)^+$ is generated by $M(1)_2^+$ if $\dim H > 1$ and V_L^+ is a direct sum of eigenspaces for $M(1)_2^+$ (cf. [AD]), the structure of $Aut(V_L^+)$ can be determined easily. We shall use a classic result.

Proposition 3.1. *The automorphism group of the Jordan algebra of symmetric $n \times n$ matrices is $PO(n,\mathbb{C})$, acting by conjugation.*

Proof. [J]. □

3.1 $Aut(M(1)^+)$

We first recall the construction of $M(1)^+$. Let H be a n-dimensional complex vector space with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ and $\hat{H} = H \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\, c$ the corresponding affine Lie algebra. Consider the induced $\hat{H}$-module

$$M(1) = \mathcal{U}(\hat{H}) \otimes_{\mathcal{U}(H\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C} \simeq S(H \otimes t^{-1}\mathbb{C}[t^{-1}]) \quad \text{(linearly)}$$

where $H \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$, and c acts as 1. For $\alpha \in H$ and $n \in \mathbb{Z}$ we set $\alpha(n) := \alpha \otimes t^n$. Let τ be the automorphism of $M(1)$ such that

$$\tau(\alpha_1(-n_1)\cdots\alpha_k(-n_k)) = (-1)^k \alpha_1(-n_1)\cdots\alpha_k(-n_k)$$

for $\alpha_i \in H$ and $n_1 \geq \cdots \geq n_k \geq 1$. Then $M(1)^+$ is the fixed point subspace of τ.

Proposition 3.2. *The automorphism group of $M(1)^+$ is $PO(n,\mathbb{C})$.*

Proof. We first deal with the case that $\dim H > 1$. Then $M(1)^+$ is generated by $M(1)^+_2$ (cf. [DN2]), which is a Jordan algebra under $u \cdot v = u_1 v$ for $u, v \in M(1)^+_2$. So any automorphism of $M(1)^+$ restricts to an automorphism of the Jordan algebra $M(1)^+_2$. On the other hand, the automorphism group of $M(1)$ is $O(n,\mathbb{C})$ [DM2], which preserves $M(1)^+$. Clearly, the kernel of the action of $O(n,\mathbb{C})$ on $M(1)^+$ is $\{\pm 1\}$. As a result $PO(n,\mathbb{C})$ is a subgroup of the automorphism group of $M(1)^+$. By Proposition 3.1, any automorphism of $M(1)^+_2$ extends to an automorphism of $M(1)^+$.

We now assume that $\dim H = 1$. Then $M(1)^+$ is not generated by $M(1)^+_2$. By Lemma 2.6 and Theorem 2.7 of [DG1] for any nonnegative even integer n there is a unique lowest weight vector u^n (up to scalar multiple) of weight n^2 and $M(1)^+$ is generated by the Virasoro vector and u^n. Using the fusion rule given in Lemma 2.6 of [DG1] we immediately see that the automorphism group of $M(1)^+$ in this case is trivial. Clearly, $PO(1,\mathbb{C}) = 1$. This finishes the proof. □

3.2 $\mathbf{Aut}(V_L^+)$

First we review from [B] and [FLM] the construction of lattice vertex operator algebra V_L for any positive definite even lattice L. Let $H = \mathbb{C} \otimes_{\mathbb{Z}} L$. Recall that $\hat{L}$ is the

canonical central extension of L by the cyclic group $\langle \pm 1\rangle$ such that the commutator map is given by $c(\alpha,\beta)=(-1)^{(\alpha,\beta)}$. We fix a bimultiplicative 2-cocycle $\epsilon : L\times L \to \langle \pm 1\rangle$ such that $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha)=c(\alpha,\beta)$ for $\alpha,\beta\in L$. Form the induced $\hat{L}$-module

$$\mathbb{C}\{L\}=\mathbb{C}[\hat{L}]\otimes_{\mathbb{C}[\langle\pm 1\rangle]}\mathbb{C}\simeq \mathbb{C}[L] \ \text{ (linearly)},$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and -1 acts on $\mathbb{C}$ as multiplication by -1. For $a\in\hat{L}$, write $\iota(a)$ for $a\otimes 1$ in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by: $a\cdot\iota(b)=\iota(ab)$ for $a,b\in\hat{L}$. If $(L,L)\subset 2\mathbb{Z}$ then $\mathbb{C}\{L\}$ and $\mathbb{C}[L]$ are isomorphic algebras. The lattice vertex operator algebra V_L is defined to be $M(1)\otimes\mathbb{C}\{L\}$, as a vector space.

Then $O(\hat{L})$ is a naturally defined subgroup of $\mathrm{Aut}(\hat{L})$ and $Hom(L,\mathbb{Z}/2\mathbb{Z})$ may be identified with a subgroup of $O(\hat{L})$ (see [FLM] , [DN1], [GH]) and there is an exact sequence

$$1\to Hom(L,\mathbb{Z}/2\mathbb{Z})\to O(\hat{L})\xrightarrow{-} O(L)\to 1.$$

It is proved in [DN1] that $Aut(V_L)$ has shape $N{\cdot}O(\hat{L})$ where N is the normal subgroup of $Aut(V_L)$ generated by e^{u_0} for $u\in(V_L)_1$. Note that $Hom(L,\mathbb{Z}/2\mathbb{Z})$ can furthermore be identified with the intersection of N and $O(\hat{L})$. See the List of Notations.

Let $e: L\to\hat{L}$ be a section associated to the 2-cocycle ϵ, written $\alpha\mapsto e_\alpha$. Let θ be the automorphism θ of $\hat{L}$ of order 2 such that $\theta e_\alpha=e_{-\alpha}$ for $\alpha\in L$. Then θ extends to an automorphism of V_L, still denoted by θ, such that $\theta|_{M(1)}$ is identified with τ and $\theta\iota(a)=\iota(\theta a)$ for all $a\in\hat{L}$. Set $e^\alpha=\iota(e_\alpha)$. Then $\theta e^\alpha=e^{-\alpha}$.

Let V_L^+ be the fixed points of θ. In order to determine the automorphism group of V_L^+, it is important to understand which automorphism of V_L restricts to an automorphism of V_L^+. Clearly, the centralizer of θ in $Aut(V_L)$ acts on V_L^+. So we get an action of $O(\hat{L})/\langle\pm 1\rangle$ on V_L^+. Let $h\in H$. Then $e^{2\pi i h(0)}$ preserves V_L^+ if and only if $(h,\alpha)\equiv(h,-\alpha)$ modulo $\mathbb{Z}$ for any $\alpha\in L$. That is, $h\in\frac{1}{2}L^*$ where L^* is the dual lattice of L.

Lemma 3.3. *The subgroup of $Aut(V_L^+)$ which preserves $M(1)_2^+$ is just the LVOA$^+$-group.*

Proof. Let $n:=dim(H)$. Let $\sigma\in Aut(V_L^+)$ such that $\sigma M(1)_2^+\subset M(1)_2^+$. Then $\sigma|_{M(1)_2^+}\in PO(n,\mathbb{C})$ as in 3.2. Note that $M(1)^+$ is generated by $M(1)_2^+$ as $rank(L)>1$ (see the proof of Proposition 3.2). So, σ preserves $M(1)^+$.

For any $\alpha\in L$, let $V_L^+(\alpha)$ be the $M(1)^+$-submodule generated by $v_\alpha:=e^\alpha+e^{-\alpha}$. Then $V_L^+(\alpha)$ is an irreducible $M(1)^+$-module, $V_L^+(\alpha)$ and $V_L^+(\beta)$ are isomorphic $M(1)^+$-modules if and only if $\alpha=\pm\beta$ (cf. [AD]). Moreover, if $\alpha\neq 0$ then $V_L^+(\alpha)$ is isomorphic to $M(1)\otimes e^\alpha$ (cf. [AD]).

Note that $V_L^+ = \sum_{\alpha\in L} V_L^+(\alpha)$. Let S be a subset of L such that $|S\cap\{\pm\alpha\}| = 1$ for any $\alpha \in L$. Then for any two different $\alpha, \beta \in S$, $V_L^+(\alpha)$ and $V_L^+(\beta)$ are nonisomorphic $M(1)^+$-modules and

$$V_L^+ = \oplus_{\alpha\in S} V_L^+(\alpha)$$

is a direct sum of nonisomorphic irreducible $M(1)^+$-modules.

Let $\alpha \in L$. Since σ preserves $M(1)^+$, it sends $V_L^+(\alpha)$ to $V_L^+(\beta)$ for some $\beta \in L$. The vector v_α is the unique lowest weight vector (up to a scalar) of $V_L^+(\alpha)$. This implies that $\sigma(v_\alpha) = \lambda v_\beta$ for some nonzero scalar $\lambda \in \mathbb{C}$ (depending on α and β).

For a vertex operator algebra V and a homogeneous $v \in V$ we set $o(v) = v_{\mathrm{wt}v-1}$ and extend to all of V linearly. Note that v_α is an eigenvector for $o(v)$ for $v \in M(1)_2^+$. In fact, $o(h_1(-1)h_2(-1))v_\alpha = (h_1,\alpha)(h_2,\alpha)v_\alpha$ for $h_i \in H$. Recall the proof of Proposition 3.2. We can regard the restriction of σ to $(V_L^+)_2 \cong M(1)_2^+$ as an element of $O(n,\mathbb{C})$, well-defined modulo ± 1. Then $\sigma(h_1(-1)h_2(-1)) = (\sigma h_1)(-1)(\sigma h_2)(-1)$. Note that σ^{-1} is the adjoint of σ. Then,

$$\begin{aligned}&(h_1,\alpha)(h_2,\alpha)\lambda v_\beta = \sigma((h_1,\alpha)(h_2,\alpha)v_\alpha) = \sigma(o(h_1(-1)h_2(-1))v_\alpha)\\ &= o(\sigma(h_1(-1)h_2(-1)))\lambda v_\beta = (\sigma h_1,\beta)(\sigma h_2,\beta)\lambda v_\beta.\end{aligned}$$

Since the h_i are arbitrary, $\sigma\alpha = \pm\beta$. Thus σ maps L onto L so induces an isometry of L which is well defined modulo $\langle\pm 1\rangle$.

Multiplying σ by an element from $\mathrm{LVOAG}^+(L)$ (which comes from $\mathbb{N}(\hat{L})$), we can assume that $\sigma|_{M(1)^+} = id_{M(1)^+}$ Then $\sigma v_\alpha = \lambda_\alpha v_\alpha$ for some nonzero $\lambda_\alpha \in \mathbb{C}$. Since $V_L^+(\alpha)$ is an irreducible $M(1)^+$-module we see that σ acts as the scalar λ_α on $V_L^+(\alpha)$. Clearly, $\lambda_\alpha = \lambda_{-\alpha}$. Note that

$$\begin{aligned}&Y(v_\alpha,z)v_\beta = E^-(-\alpha,z)\epsilon(\alpha,\beta)e^{\alpha+\beta}z^{(\alpha,\beta)} + E^-(-\alpha,z)\epsilon(\alpha,-\beta)e^{\alpha-\beta}z^{-(\alpha,\beta)}\\ &+E^-(\alpha,z)\epsilon(-\alpha,\beta)e^{-\alpha+\beta}z^{-(\alpha,\beta)} + E^-(\alpha,z)\epsilon(\alpha,\beta)e^{-\alpha-\beta}z^{(\alpha,\beta)}\end{aligned}$$

where

$$E^-(\alpha,z) = \exp(\sum_{n<0}\frac{\alpha(n)z^{-n}}{n}).$$

Thus, if n is sufficiently negative, $(v_\alpha)_n(v_\beta) = u+v$ for some nonzero $u \in V_L^+(\alpha+\beta)$ and $v \in V_L^+(-\alpha+\beta)$. This gives $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta} = \lambda_{\alpha-\beta}$ by applying σ to $(v_\alpha)_n(v_\beta) = u+v$. So $\alpha \mapsto \lambda_\alpha$ defines a character of abelian group $L/2L$ of order 2^n. Clearly, any character $\lambda : L/2L \to \langle\pm 1\rangle$ defines an automorphism σ which acts on $V_L^+(\alpha)$ as λ_α. As a result, the subgroup of $Aut(V_L^+)$ which acts trivially on $M(1)^+$ is isomorphic the dual group of $L/2L$ and is exactly the subgroup of $O(\hat{L})/\langle\pm 1\rangle$ which we identified

as $\mathrm{Hom}(L, \mathbb{Z}/2\mathbb{Z})$. As a result, the subgroup of $Aut(V_L^+)$ which preserves $M(1)_2^+$ is exactly the group $O(\hat{L})/\langle \pm 1\rangle$, as desired. $\square$

Proposition 3.4. *Let L be a positive definite even lattice such that $L_1 = L_2 = \emptyset$. Then $Aut(V_L^+)$ is the inherited group, i.e., the LVOA$^+$-group.*

Proof. In this case we have $(V_L^+)_2 = M(1)_2^+$. Thus, any automorphism of V_L^+ preserves $M(1)_2^+$. By Lemma 3.3, $Aut(V_L^+)$ is the LVOA$^+$-group. $\square$

4 Rank 2 lattices

All lattices in this article are positive definite. Throughout this article, L denotes an even integral lattice. We recall a general result.

Lemma 4.1. *Let L be a lattice and M a sublattice.*
(i) If $|L : M|$ is finite, $det(M) = det(L)|L : M|^2$.
(ii) If M is a direct summand of L, $L/[M + ann_L(M)]$ embeds in $\mathcal{D}(M)$.

Proof. These are standard results. For example, see [G3]. $\square$

We need to sort out rank 2 lattices by whether they contain roots or elements of order 4, due to their contributions to low degree terms of the lattice VOA. We shall use the notations 2.1.

Lemma 4.2. Suppose that $rank(L_1) = 2$. Then L_1 spans L and L is one of $L_{A_1^2}$ or L_{A_2}.

Proof. The span of L_1 is isometric to $L_{A_1^2}$ or L_{A_2}. Each of these is a maximal even integral lattice under containment. $\square$

Lemma 4.3. Suppose that $rank(L_1) = 1$. Let $r \in L_1$ and let s generate $ann_L(r)$. Then $(s, s) \geq 4$ and if $L > span\{r, s\}$, then $14 \leq (s, s) \in 6 + 8\mathbb{Z}$.

Proof. Note that $\mathbb{Z}r$ is a direct summand of L. We have $(s, s) \geq 4$. In case $L > N := span\{r, s\}$, L/N has order 2, by 4.1. If x represents the nontrivial coset, $(x, x) \geq 4$ then $(2x, 2x) \geq 16$. Also, $(2x, 2x) \in 8\mathbb{Z}$. Since (x, r) is odd, if we write $2x = pr + qs$, for integers $p, q \in \mathbb{Z}$, then p is odd and so $q^2(s, s) \in 6 + 8\mathbb{Z}$. It follows that q is odd and $(s, s) \in 6 + 8\mathbb{Z}$. $\square$

Lemma 4.4. *Suppose that $L_1 = \emptyset$ and $rank(L_2) = 2$. If r, s are linearly independent norm 4 elements, then they span L and have Gram matrix $G = \begin{pmatrix} 4 & b \\ b & 4 \end{pmatrix}$, for some $b \in \{0, \pm 1, \pm 2\}$.*

Proof. If $L \neq N := span\{r, s\}$, then $det(N) = 16 - b^2$ is divisible by a perfect square, whence $b = 0$ or $b = \pm 2$ and the index is 2. Actually, $b = 0$ does not occur here since $\frac{1}{2}r, \frac{1}{2}s \notin L$ implies that $\frac{1}{2}(r+s) \in L_1$, a contradiction. So, $b = \pm 2$. Clearly, $span\{r, s\} \cong \sqrt{2}L_{A_2}$. However, any integral lattice containing the latter with index 2 is odd, a contradiction. Therefore, $L = N$ and the Gram matrix is as above. Positive definiteness implies that $|b| < 4$ and rootlessness implies that $b \neq \pm 3$. $\square$

Lemma 4.5. *Suppose that $L_1 = \emptyset$ and $rank(L_2) = 1$. Let $r \in L_2$ and let s generate $ann_L(x)$. Then $(s, s) \geq 6$ and $L/span\{r, s\}$ is a subgroup of $\mathbb{Z}_4$.*

If the order of $L/span\{r, s\}$ is $2, 8 \leq (s, s) \in 4 + 8\mathbb{Z}$.

If the order of $L/span\{r, s\}$ is $4, 28 \leq (s, s) \in 28 + 32\mathbb{Z}$.

Proof. Let x be in a nontrivial coset of $N := span\{r, s\}$ in L.

If $(x, r) \in 2 + 4\mathbb{Z}$, $2x = pr + qs$, where p is odd. We have $(x, x) \geq 6$, p odd and $q \neq 0$. Therefore, $(2x, 2x) \in 8\mathbb{Z}$, $24 \leq 4p^2 + (s, s)q^2$, whence q is odd and $(s, s) \in 4 + 8\mathbb{Z}$.

If $(x, r) \in 1 + 2\mathbb{Z}$, $(4x, 4x) \in 32\mathbb{Z}$. We have $(x, x) \geq 6$, whence $(4x, 4x) \geq 96$. If we write $4x = pr + qs$, we have $4p^2 + q^2(s, s) \in 32\mathbb{Z}$. Since p is odd, $p^2 \in 1 + 8\mathbb{Z}$ and $4p^2 \in 4 + 32\mathbb{Z}$. Since (s, s) is even, q is odd, $q^2 \in 1 + 8\mathbb{Z}$ and $(s, s) \in 4 + 8\mathbb{Z}$. It follows that $\frac{1}{4}q^2(s, s) \in 7 + 8\mathbb{Z}$ whence $\frac{1}{4}(s, s) \in 7 + 8\mathbb{Z}$. $\square$

5 About idempotents in small dimensional algebras

We can derive a lot of information about the automorphism group of a vertex operator algebra by restricting to low degree homogeneous pieces. For the V_L^+ problem, the degree 2 piece and its product $x, y \mapsto x_1 y$ give an algebra which is useful to study. Here, for $rank(L) = 2$, we concentrate on some commutative algebras of dimension around 5. Commutativity of $(V_L^+, 1^{st})$ is implied if $L_1 = \emptyset$, which is so for $b \in \{0, \pm 1, \pm 2\}$ as in 4.4.

It does not seem advantageous to give particular values to b most of the time, so we keep it as an unspecified constant in case these arguments might be a model for future work. In the present work, we shall note limits on b, as needed.

Notation 5.1. Let S be the Jordan algebra of degree 2 symmetric matrices and suppose that A is a commutative 5 dimensional algebra of the form $A = S \oplus \mathbb{C}v_r \oplus \mathbb{C}v_s$. Suppose that $v_r \times v_s = 0$ and that the notations of Appendix: Algebraic rules apply here, with the usual inner products and algebra product. Let $w = p + c_r v_r + c_s v_s$ be

an idempotent. Suppose also that t is a norm 4 vector orthogonal to r. Let a_1, a_2, a_3 be scalars so that $p = a_1 r^2 + a_2 rt + a_3 t^2$.

Remark 5.2. We note that the basis r, s of H has dual basis r^*, s^*, where $r^* = \frac{1}{16-b^2}(4r - bs)$ and $s^* = \frac{1}{16-b^2}(4s - br)$. The identity of A is $\frac{1}{4}\frac{1}{16-b^2}(rr^* + ss^*) = \frac{1}{4}\frac{1}{16-b^2}(4r^2 + 4s^2 - 2brs)$.

Notation 5.3. If w is an element of A, write $w = p+q$ for $p \in S^2H$ and $q \in \mathbb{C}v_r \oplus \mathbb{C}v_s$. Call the element $\bar{w} := p - q$ the *conjugate element*. The components p, q are called the *P-part* and the *Q-part* of w. Extend this notation to subscripted elements: $w_i = p_i + q_i$, $\bar{w}_i = p_i - q_i$, for indices i.

Remark 5.4. In 5.3, $q^2 \in S^2H$ since $v_r \times v_s = 0$. Also $w = p + q$ is an idempotent if and only if $p = p^2 + q^2$ and $q = 2p \times q$. Therefore, $w = p + q$ is an idempotent if and only if the conjugate $p - q$ is an idempotent.

Lemma 5.5. Suppose that w_1 and w_2 are idempotents and their sum is an idempotent. Then $w_1 \times w_2 = 0$ and $(w_1, w_2) = 0$.

Proof. We have $(w_1 + w_2)^2 = w_1^2 + 2w_1 \times w_2 + w_2^2$, whence $w_1 \times w_2 = 0$. Also, $(w_1, w_2) = (w_1^2, w_2) = (w_1, w_1 \times w_2) = 0$. $\square$

Definition 5.6. Throughout this article, an idempotent is not zero or the identity, unless the context clearly allows the possibility. We call an idempotent w of *type 0, 1, 2*, respectively, if it has Q-part which is 0, is a multiple of v_r or v_s, or is not a multiple of either v_r or v_s.

Lemma 5.7. Then (i) $r^2 \times s^2 = 4brs$, $r^2 \times r^2 = 16r^2$, $s^2 \times s^2 = 16s^2$, $rs \times rs = 4r^2 + 4s^2 + 2brs$; $x^2 \times v_r = (x, r)^2 v_r = \frac{1}{2}(x^2, r^2)v_r$; $r^2 \times rs = 8rs + 2b^2r^2$, $s^2 \times rs = 8rs + 2b^2s^2$; also $v_r \times v_r = r^2, v_r \times v_s = 0, v_s \times v_s = s^2$.

(ii) $(r^2, r^2) = 32 = (s^2, s^2)$, $(r^2, s^2) = 2b^2$, $(rs, rs) = 16 + b^2$, $(rs, r^2) = 8b = (rs, s^2)$ and $(v_r, v_r) = 2 = (v_s, v_s)$ and $(v_r, v_s) = 0$.

Proof. See the Appendix (and take $a = d = 4$, $b \neq 0, \pm 2$). $\square$

5.1 Idempotents of type 0

Lemma 5.8. These are just idempotents in the Jordan algebra of symmetric matrices. They are ordinary idempotent matrices which are symmetric. Up to conjugacy by orthogonal transformation, they are diagonal matrices with diagonal entries only 1 and 0.

5.2 Idempotents of type 1

Notation 5.9. The next few results apply to the case of an idempotent of type 1, i.e., the form $w = p + c_r v_r$, where $c_r \neq 0$. In such a case, $w = w^2 = p^2 + c_r^2 r^2 + c_r(p, r^2)v_r$ (see the Appendix : Algebraic Rules) From $c_r \neq 0$, we get $(p, r^2) = 1$. We continue to use the notation of 5.1.

Lemma 5.10. *Suppose that $c_r \neq 0$ and $c_s = 0$. We have $a_1 = 16a_1^2 + 4a_2^2 + c_r^2$; $a_2 = 16a_2(a_1 + a_3)$; $a_3 = 16a_3^2 + 4a_2^2$ and $(p, r^2) = 1$.*

Proof. Compute $p + c_r v_r = w = w^2 = p^2 + c_r^2 r^2 + c_r(p, r^2)v_r$ (see 5.7) and expand in the basis r^2, rt, t^2, v_r. □

Corollary 5.11. $a_1 = \frac{1}{32}$.

Proof. We have $1 = (p, r^2) = a_1(r^2, r^2) = 32a_1$, whence $a_1 = \frac{1}{32}$. □

Lemma 5.12. *Suppose that $c_r \neq 0$ and $c_s = 0$. Then*
(A1) $a_1 = 16a_1^2 + 4a_2^2 + c_r^2$;
(A2) $a_2 = 16a_2(a_1 + a_3)$; and
(A3) $a_3 = 16a_3^2 + 4a_2^2$.

Proof. Compute $p^2 = (16a_1^2 + 4a_2^2)r^2 + 16(a_1 a_2 + a_3 a_2)rt + (16a_3^2 + 4a_2^2)t^2$ and use $w = w^2 = p^2 + c_r^2 r^2 + c_r v_r$. □

Lemma 5.13. *Suppose that $c_r \neq 0$ and $c_s = 0$. If $a_2 = 0$, then $a_3 \in \{0, \frac{1}{16}\}$ and $c_r = \pm\frac{1}{8}$.*

Proof. We deduce from (A3) that $a_3 = 16a_3^2$, then $c_r = \pm\frac{1}{8}$. □

Lemma 5.14. *Suppose that $c_r \neq 0$ and $c_s = 0$. Then $a_2 = 0$.*

Proof. If $a_2 \neq 0$, then from (A2), $1 = 16(a_1 + a_3)$ and we get $a_3 = \frac{1}{32}$. Next, use (A3) to get $\frac{1}{64} = 4a_2^2$. Finally use (A1) to get $c_r = 0$. □

Theorem 5.15. *Assume that $c_r \neq 0$ and $c_s = 0$. Then*
(i) $a_1 = \frac{1}{32}, a_2 = 0, c_r = \pm\frac{1}{8}$; and
(ii) either $(w, w) = \frac{1}{16}$ and $a_3 = 0$; or $(w, w) = \frac{3}{16}$ and $a_3 = \frac{1}{16}$.
All of the above cases occur. If an idempotent occurs, so does its complementary idempotent.

Proof. This is a summary of preceding results. □

Lemma 5.16. *If w is an idempotent of type 1, then*

(i) if $(w, w) = \frac{1}{16}$, the eigenvalues for $ad(w)$ are $1, 0, 0, \frac{1}{4}, \frac{b^2}{32}$. Eigenvectors for these respective eigenspaces are $w, 1 - w, t^2, rt, v_s$;

(ii) if $(w, w) = \frac{3}{16}$, the eigenvalues for $ad(w)$ are $0, 1, 1, \frac{3}{4}, 1 - \frac{b^2}{32}$. Eigenvectors for these respective eigenspaces are $w, 1 - w, t^2, rt, v_s$.

If $\frac{b^2}{32} \neq 0, 1, \frac{1}{4}, \frac{3}{4}$, the multiplicities of 0 and 1 are 2 and 1 in case (i) and 1 and 2 in case (ii).

Proof. Straightforward calculation. Note that $\frac{b^2}{32} \neq 0, 1, \frac{1}{4}, \frac{3}{4}$ follows if b is rational $\square$

Corollary 5.17. *If w is a type 1 idempotent and is the sum of two nonzero idempotents, w_1, w_2, then w has the form $\frac{1}{32}r^2 + \frac{1}{16}t^2 \pm \frac{1}{8}v_r$ and w_1, w_2 are, up to order, $\frac{1}{32}r^2 \pm \frac{1}{8}v_r$ and $\frac{1}{16}t^2$.*

Proof. If w is such a sum, each w_i is in the 1-eigenspace of $ad(w)$, which must be more than 1-dimensional. This means that w has norm $\frac{3}{16}$ and one of the w_i, say for $i = 1$, has type 1 and Q-part $\pm\frac{1}{8}v_r$. Therefore, w_2 has type 0, whence norm $\frac{1}{8}$. This means that w_1 has norm $\frac{1}{16}$ and so we know that w_1 has shape $\frac{1}{32}r^2 \pm \frac{1}{8}v_r$ and $w = \frac{1}{16}t^2$. $\square$

5.3 Idempotents of type 2

Hypothesis 5.18. We assume in this subsection that the parameter $b \neq 0, \pm 2, \pm 3$ (which means $b = \pm 1$). Then the algebra $(V_2, 1^{st})$ is commutative since $V_1 = 0$.

Notation 5.19. $p = c(r^2 + s^2) + drs$, $v = c_r v_r + c_s v_s$.

Lemma 5.20. *If c_r and c_s are nonzero, then there are at most 8 possibilities for w. In more detail, there are at most two values of c (and, correspondingly, of d). We have $c_r^2 = c_s^2$ and this common value depends on c (or on d).*

Proof. Compute $p + c_r v_r + c_s v_s = w = w^2 = p^2 + c_r^2 r^2 + c_s^2 s^2 + c_r (p, r^2) v_r + c_s (p, s^2) v_s$. Since c_r and c_s are nonzero, $(p, r^2) = 1 = (p, s^2)$.

Since $(r^2, r^2) = 32 = (s^2, s^2)$, $(rs, r^2) = 8b = (rs, s^2)$ and $(r^2, s^2) = 16 + b^2$, we have $p = c(r^2 + s^2) + drs$. for some scalars, c, d. The previous paragraph then implies that $1 = (32 + 2b^2)c + 8bd$. Since $b \neq 0$,

$$(e1) \qquad d = \frac{1}{8b}(2c(32 + 2b^2) - 1)$$

is a linear expression in c.

Now, $p^2 = (16c^2+4d^2)(r^2+s^2)+(2bc^2+2bd^2)rs$ and so $w^2 = (16c^2+4d^2+c_r^2)r^2 + (16c^2+4d^2+c_s^2)s^2+(2bc^2+2bd^2)rs+c_rv_r+c_sv_s$. It follows that $c_r^2 = c_s^2$.

We compare coefficients of r^2 and get

$$c = 16c^2 + 4d^2 + c_r^2. \tag{e2}$$

We compare coefficients of rs and get

$$d = 2bc^2 + 2bd^2. \tag{e3}$$

Since d is a linear expression in c, c satisfies a quadratic equation, depending on b but not $c_r^2 = c_s^2$. The degree of this equation really is 2 since $b \neq 0$ real implies that the top coefficient is nonzero.

It follows that the ordered pair d, c has at most two possible values. For each, there is a unique value for c_r^2, hence at most two possible values for c_r (and the same two for c_s). Therefore there are at most eight idempotents of type 2. □

Lemma 5.21. *$c \neq 0$ and $d \neq 0$.*

Proof. Suppose that $c = 0$. We then have $p = drs$, $d = \frac{-1}{8b}$. On the other hand, since $w = drs + v$ is an idempotent, the coefficient for w^2 at rs is $d = 8bc^2 + 2bd^2 = 2bd^2 = 2bd^2$, which implies that $1 = 2bd$. This is incompatible with $d = \frac{-1}{8b}$.

If $d = 0$, equation (e3) implies that $c = 0$, which is false. □

Lemma 5.22. *If w is a type 2 idempotent, $w = c(r^2+s^2) + drs + c_rv_r + c_sv_s$ and $1 - w$ is the complementary idempotent, expanded similarly as $1 - w = c'(r^2+s^2) + d'rs + c'_rv_r + c'_sv_s$, then $c'_r = -c_r \neq c_r$, $c'_s = -c_s$, $c \neq c'$ and $d \neq d'$. In particular, in the notation of 5.20, the function $c \mapsto c_r^2$ is 2-to-1 and so only one value of c_r^2 occurs for type 2 idempotents.*

Proof. If it were true that $c = c'$, then $w = \frac{1}{2}\mathbb{I} + v$ and $1 - w = \frac{1}{2}\mathbb{I} - v$. Since these are idempotents, $v^2 = \frac{1}{2}\mathbb{I}$. However, this is impossible as $b \neq \pm 2$ implies that v^2 is a multiple of $r^2 + s^2$ and $\mathbb{I}$ is not a linear combination of r^2, s^2 for $b \neq 0$ (see 5.2). □

5.4 Sums of idempotents

Hypothesis 5.23. We continue to take $b = \pm 1$. Results of the previous subsection apply.

In the arguments in this section, we allow the symbol b to be any odd integer, though the lattice is positive definite only for $b = \pm 1$.

Lemma 5.24. *Suppose that w_1, w_2 are two idempotents of type 1. If $w_1 + w_2$ is an idempotent, then $w_1 + w_2$ does not have type 1 or type 2.*

Proof. We eliminate the sum having type 1 with Lemmas 5.17. To eliminate a sum having type 2, we note that for type 1 idempotents, we have $a_2 = 0$ by 5.15, whereas $d \neq 0$ for type 2, by 5.21. □

Lemma 5.25. *If w_1, w_2 are idempotents of type 2 and not complementary, their sum is not an idempotent.*

Proof. Assume that the sum w is an idempotent. From 5.24, the sum has type 0, so has the form $\frac{1}{16}u^2$, for some vector $u \in H$ of norm 4. The eigenvalues of $ad(u)$ are $1, 0, \frac{1}{2}$ and $\frac{1}{16}(u,r)^2$, $\frac{1}{16}(u,s)^2$, with respective eigenvectors $u^2, 1-u^2, \frac{1}{2}uu', v_r, v_s$, where u' spans the orthogonal of u in H.

Now, w_1, w_2 are linearly independent (or else they are equal, which is impossible). This means that the eigenvalue 1 has multiplicity at least 2. So, at least one of $(u,r)^2, (u,s)^2$ is 16. Since w_1, w_2 lie in the 1-eigenspace of $ad(u)$ and both w_i have type 2, both these square norms must be 16, i.e., $m := (u,r) = \pm 4$ and $n := (u,s) = \pm 4$. Since r, s form a basis and the form is nonsingular, this forces $u = mr^* + ns^*$, where r^*, s^* is the dual basis. We have $4 = (u,u) = 16(r^*, r^*) + 2mn(r^*, s^*) + 16(s^*, s^*)$. The right side is $\frac{1}{16-b^2}[16(4r-bs, 4r-bs) + 2mn(4r-bs, 4s-br) + 16(4s-br, 4s-br)]$.

Since b is an odd integer, the above rational number in reduced form clearly has numerator divisible by 16, so does not equal 4, a contradiction. □

Lemma 5.26. *The sum of a type 1 and type 2 idempotent is not an idempotent.*

Proof. Assume that $w := w_1 + w_2$ is an idempotent. Obviously it does not have type 0. By 5.17, it does not have type 1.

We conclude that w has type 2. However, the coefficients of w at r^2 and s^2 must be equal for type 2, a contradiction since this forces the P-part if the type 1 idempotent to be 0. □

Corollary 5.27. *The only idempotents which are a proper summand of some nontrivial idempotent are the ones of type 1 and norm $\frac{1}{16}$. There are 4 such and they come in orthogonal pairs, which are just pair of idempotents and their conjugates.*

Corollary 5.28. *$Aut(A)$ is a dihedral group of order 8.*

Proof. The automorphism group preserves and acts faithfully on the set J of type 1 idempotents of norm $\frac{1}{16}$, the complete set of idempotents which are proper summands of proper idempotents, and furthermore preserves the partition defined by

orthogonality. The orthogonal in A of the nonsingular subspace $span(J)$ is spanned by $v := r^2+s^2-\frac{16+b^2}{4b}rs$. We claim that if an automorphism acts trivially on $span(J)$, it acts trivially on A. This is so because $rs \in span\{r^2 \times s^2, r^2, s^2\}$ and $\{rs\} \cup J$ spans A.

This proves that the automorphism group of A embeds in a dihedral group of order 8. This embedding is an isomorphism onto since the LVOA$^+$- group embeds in $Aut(A)$. $\square$

Proposition 5.29. *$Aut(V_L^+)$ is just the LVOA$^+$-group, isomorphic to Dih_8.*

Proof. In this case we have $(V_L^+)_2 = M(1)_2^+$. Thus any automorphism of V_L^+ preserves $M(1)_2^+$. Now use 3.3. $\square$

6 Automorphism group of V_L^+ with $rank(L) = 2$

In this section, we assume that the rank of L is equal to 2. If $L_1 = L_2 = \emptyset$, the automorphism group of V_L^+ was determined in Proposition 3.4. So in this section we assume that L_1 or L_2 is not empty.

6.1 $L_1 = \emptyset$ and $rank(L_2) = 2$; $b = 0$.

Note that L is generated by L_2. We will discuss the automorphism group according to the value b in the Gram matrix G (see 4.4).

First we assume that $b = 0$ in the Gram matrix G. Then $L \cong \sqrt{2}L_{A_1} \perp \sqrt{2}L_{A_1}$, where L_{A_1} is the root lattice of type A_1. Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ with $(\alpha_i, \alpha_j) = 4\delta_{ij}$ for $i, j = 1, 2$. Set

$$\omega_1 = \frac{1}{16}\alpha_1(-1)^2 + \frac{1}{4}(e^{\alpha_1} + e^{-\alpha_1}),$$
$$\omega_2 = \frac{1}{16}\alpha_2(-1)^2 - \frac{1}{4}(e^{\alpha_2} + e^{-\alpha_2}).$$

We also use α_2 to define ω_3 and ω_4 in the same fashion. Then ω_i for $i = 1, 2, 3, 4$ are commutative Virasoro vectors of central charge $\frac{1}{2}$ (see [DMZ] and [DGH]). It is well-known that $(V_L^+)_2$ is a commutative (nonassociative) algebra under $u \times v = u_1 v$ since the degree 1 part is 0 (cf. [FLM]). Let X be the span of ω_i for all i.

Lemma 6.1. *If $u \in (V_L)_2$ is a Virasoro vector of central charge $1/2$ then $u = \omega_i$ for some i.*

Proof. The space $(V_L)_2$ is 5-dimensional with a basis

$$\{\omega_1, \omega_2, \omega_3, \omega_4, \alpha_1(-1)\alpha_2(-1)\}.$$

Let $u = \sum_{i=1}^4 c_i\omega_i + x\alpha_1(-1)\alpha_2(-1) \in (V_L)_2$ be a Virasoro vector of central charge $1/2$. Then $u \times u = 2u$. Note that $\omega_i \times \omega_j = \delta_{i,j}2\omega_i$ for $i, j = 1, 2, 3, 4$, $\omega_i \times \alpha_1(-1)\alpha_2(-1) = \frac{1}{2}\alpha_1(-1)\alpha_2(-1)$ and $\alpha_1(-1)\alpha_2(-1) \times \alpha_1(-1)\alpha_2(-1) = 4\alpha_1(-1)^2 + 4\alpha_2(-1)^2$. So we have a nonlinear system

$$2c_i = 2c_i^2 + 32x^2, \ i = 1, 2, 3, 4$$
$$2x = \sum_{i=1}^4 xc_i.$$

If $x \neq 0$ then $\sum_{i=1}^4 c_i = 2$ and $2 = \sum_{i=1}^4 c_i^2 + 64x^2$.

Since the central charge of u is $1/2$ we have

$$\frac{1}{4} = u_3u = \sum_{i=1}^4 \frac{c_i^2}{4} + 16x^2$$

and $1 = \sum_{i=1}^4 c_i^2 + 64x^2$. This is a contradiction. So $x = 0$. This implies that $c_i = 0, 1$ and $u = \omega_i$ for some i. $\square$

By Lemma 6.1, any automorphism σ of V_L^+ induces a permutation of the four ω_i.

It is known from [FLM] that $(V_L^+)_2$ has a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ given by $(u, v) = u_3v$ for $u, v \in (V_L^+)_2$. The orthogonal complement of X in $(V_L^+)_2$ with respect to the form is spanned by $\alpha_1(-1)\alpha_2(-1)$. Thus $\sigma\alpha_1(-1)\alpha_2(-1) = \lambda\alpha_1(-1)\alpha_2(-1)$ for some nonzero constant λ. Since $\alpha_1(-1)\alpha_2(-1) \times \alpha_1(-1)\alpha_2(-1) = \alpha_1(-1)^2 + \alpha_2(-1)^2$ which is a multiple of the Virasoro element ω. This shows that $\lambda = \pm 1$.

On the other hand,

$$V_L^+ \cong V_{\sqrt{2}L_{A_1}}^+ \otimes V_{\sqrt{2}L_{A_1}}^+ \oplus V_{\sqrt{2}L_{A_1}}^- \otimes V_{\sqrt{2}L_{A_1}}^-.$$

By Corollary 3.3 of [DGH],

$$V_L^+ \cong L(\frac{1}{2}, 0)^{\otimes 4} \oplus L(\frac{1}{2}, \frac{1}{2})^{\otimes 4}.$$

So if the restriction of σ to X is identity, then the action of σ on $V_{\sqrt{2}L_{A_1}}^+ \otimes V_{\sqrt{2}L_{A_1}}^+$ is trivial and on $V_{\sqrt{2}L_{A_1}}^- \otimes V_{\sqrt{2}L_{A_1}}^-$ is ± 1. Indeed, there is automorphism τ of V_L^+ such that τ acts trivially on $V_{\sqrt{2}L_{A_1}}^+ \otimes V_{\sqrt{2}L_{A_1}}^+$ and acts as -1 on $V_{\sqrt{2}L_{A_1}}^- \otimes V_{\sqrt{2}L_{A_1}}^-$ by the fusion

role for $V^+_{\sqrt{2}L_{A_1}}$ (see [ADL]). As $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ is generated by ω_i for $i = 1, 2, 3, 4$, any automorphism preserves $V^+_{\sqrt{2}L_{A_1}} \otimes V^+_{\sqrt{2}L_{A_1}}$ and its irreducible module $V^-_{\sqrt{2}L_{A_1}} \otimes V^-_{\sqrt{2}L_{A_1}}$ (cf. [DM1]). As a result, $\langle \tau \rangle$ is a normal subgroup of $Aut(V_L^+)$ isomorphic to $\mathbb{Z}_2$.

Next we show how Sym_4 can be realized as a subgroup of $Aut(V_L^+)$ by showing that any permutation $\sigma \in Sym_4$ gives rise to an automorphism of V_L^+. But it is clear that Sym_4 acts on V_L^+ by permuting the tensor factors. In order to see that Sym_4 acts on V_L^+ as automorphisms, it is enough to show that $\sigma(Y(u, z)v) = Y(\sigma u, z)\sigma v$ for $\sigma \in Sym_4$ and $u, v \in V_L^+$. There are 4 different ways to choose u, v. We only discuss the case that $u, v \in L(\frac{1}{2}, \frac{1}{2})^{\otimes 4}$ since the other cases can be dealt with in a similar fashion. Let $u = u^1 \otimes u^2 \otimes u^3 \otimes u^4$ and $v = v^1 \otimes v^2 \otimes v^3 \otimes v^4$ where u_i, v_i are tensor factors in the i-th $L(\frac{1}{2}, \frac{1}{2})$. Let $\mathcal{Y}$ be a nonzero intertwining operator of type $\binom{L(\frac{1}{2},0)}{L(\frac{1}{2},\frac{1}{2})\ L(\frac{1}{2},\frac{1}{2})}$. Then, up to a constant,

$$Y(u, z)v = \mathcal{Y}(u_1, z)v_1 \otimes \mathcal{Y}(u_2, z)v_2 \otimes \mathcal{Y}(u_3, z)v_3 \otimes \mathcal{Y}(u_4, z)v_4$$

(see [DMZ]). Since σ is a permuation, it is trivial to verify that $\sigma(Y(u, z)v) = Y(\sigma u, z)\sigma v$.

So we have proved the following:

Proposition 6.2. *If $b = 0$ in the Gram matrix G then $L \cong \sqrt{2}L_{A_1} \times \sqrt{2}L_{A_1}$ and $Aut(V_L^+) \cong Sym_4 \times \mathbb{Z}_2$.*

Remark 6.3. Here is a different proof that $Aut(V_L^+)$ contains a copy of $Sym_4 \times \mathbb{Z}_2$, using the theory of finite subgroups of Lie groups. Our lattice L lies in $M \cong L_{A_1^2}$. Take V_M, which is a lattice VOA. By [DN1] , V_M has automorphism group isomorphic to $PSL(2, \mathbb{C}) \wr 2$. In $PSL(2, \mathbb{C})$, there is up to conjugacy a unique four group and its normalizer is isomorphic to Sym_4. Correspondingly, in $PSL(2, \mathbb{C}) \wr 2$ there is a subgroup isomorphic to $Sym_4 \wr 2$. In this, take a subgroup H of the form $2^4{:}[Sym_3 \times 2]$. Let t be an involution of H which maps to the central involution of $H/O_2(H) \cong Sym_3 \times 2$ and take $R := C_{O_2(H)}(t) \cong 2^2$. Take the fixed points V_M^R. We have that V_M^R is isomorphic to our V_L^+. So, V_L^+ gets an action of $H/R \cong 2^2{:}[Sym_3 \times 2] \cong Sym_4 \times 2$.

6.2 $L_1 = \emptyset$ and $rank(L_2) = 2$; $b = 2$.

Next we assume that b in the Gram matrix is 2. Then $L \cong \sqrt{2}L_{A_2}$. Then $L = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ with $(\alpha_i, \alpha_i) = 4$ and $(\alpha_1, \alpha_2) = 2$. As before we define ω_1, ω_2 by using α_1, ω_3, ω_4 by using α_2 and ω_5, ω_6 by using $\alpha_1 + \alpha_2$. Then ω_i, for $i = 1, ..., 6$ form a basis of $(V_L^+)_2$.

Lemma 6.4. *If $u \in (V_L^+)_2$ is a Virasoro vector of central charge $1/2$, then $u = \omega_i$ for some i.*

Proof. *First proof.* (There will be a second proof in the next section.)

Let $u = \sum_{i=1}^{6} c_i\omega_i$ for some $c_i \in \mathbb{C}$. Then u is a Virasoro vector of central charge $1/2$ if and only if $(u,u) = 1/4$ and $u \times u = 2u$. Note that

$$(\omega_i, \omega_i) = 1/4, (\omega_{2j-1}, \omega_{2j}) = 0, \quad 1 \le i \le 6, j = 1,2,3$$

$$(\omega_1, \omega_k) = (\omega_2, \omega_k) = \frac{1}{32}, \quad k = 3,4,5,6.$$

So we have

$$(u,u) = \frac{1}{4}\sum_{i=1}^{6} c_i^2 + \frac{1}{16}\sum_{j=1,2}\sum_{j<k\le 3}(c_{2j-1}c_{2k-1} + c_{2j-1}c_{2k} + c_{2j}c_{2k-1} + c_{2j}c_{2k}) = \frac{1}{4}.$$

In order to compute $u \times u$ we need the following multiplication table in $(V_L^+)_2$:

$$\omega_{2i-1} \times \omega_{2i} = 0, i = 1,2,3$$

$$\omega_1 \times \omega_3 = \frac{1}{4}(\omega_1 + \omega_3 - \omega_6), \quad \omega_2 \times \omega_3 = \frac{1}{4}(\omega_2 + \omega_3 - \omega_5)$$

$$\omega_1 \times \omega_4 = \frac{1}{4}(\omega_1 + \omega_4 - \omega_5), \quad \omega_2 \times \omega_4 = \frac{1}{4}(\omega_2 + \omega_4 - \omega_6)$$

$$\omega_1 \times \omega_5 = \frac{1}{4}(\omega_1 + \omega_5 - \omega_4), \quad \omega_2 \times \omega_5 = \frac{1}{4}(\omega_2 + \omega_5 - \omega_3)$$

$$\omega_1 \times \omega_6 = \frac{1}{4}(\omega_1 + \omega_6 - \omega_3), \quad \omega_2 \times \omega_6 = \frac{1}{4}(\omega_2 + \omega_6 - \omega_4)$$

$$\omega_3 \times \omega_5 = \frac{1}{4}(\omega_3 + \omega_5 - \omega_2), \quad \omega_4 \times \omega_5 = \frac{1}{4}(\omega_4 + \omega_5 - \omega_1)$$

$$\omega_3 \times \omega_6 = \frac{1}{4}(\omega_3 + \omega_6 - \omega_1), \quad \omega_2 \times \omega_6 = \frac{1}{4}(\omega_2 + \omega_6 - \omega_2).$$

Then $u \times u = 2u$ if and only if

$$c_1^2 + \frac{1}{4}(c_1c_3 + c_1c_4 + c_1c_5 + c_1c_6 - c_3c_6 - c_4c_5) = c_1$$

$$c_2^2 + \frac{1}{4}(c_2c_3 + c_2c_4 + c_2c_5 + c_2c_6 - c_3c_5 - c_4c_6) = c_2$$

$$c_3^2 + \frac{1}{4}(c_1c_3 + c_2c_3 + c_3c_5 + c_3c_6 - c_1c_6 - c_2c_5) = c_3$$

$$c_4^2 + \frac{1}{4}(c_1c_4 + c_2c_4 + c_4c_5 + c_4c_6 - c_1c_5 - c_2c_6) = c_4$$
$$c_5^2 + \frac{1}{4}(c_1c_5 + c_2c_5 + c_3c_5 + c_4c_6 - c_1c_4 - c_2c_3) = c_5$$
$$c_6^2 + \frac{1}{4}(c_1c_6 + c_2c_6 + c_3c_6 + c_4c_6 - c_1c_3 - c_2c_4) = c_6.$$

There are exactly 6 solutions to this linear system: $c_i = 1$ and $c_j = 0$ if $j \neq i$ where $i = 1, ..., 6$. We thank Harm Derksen for obtaining this result with the MacCauley software package. This finishes the proof of the lemma. $\square$

Proposition 6.5. *If $b = 2$ in the Gram matrix, then $L \cong \sqrt{2}L_{A_2}$ and $Aut(V_L^+)$ is the LVOA$^+$-group.*

Proof. First note that the Weyl group acts on L, preserving and acting as Sym_3 on the set

$$\{\{\pm\alpha_1\}, \{\pm\alpha_2\}, \{\pm(\alpha_1 + \alpha_2)\}\}.$$

Now let $\sigma \in Aut(V_L^+)$. Set $X_i = \{\omega_{2i-1}, \omega_{2i}\}$ for $i = 1, 2, 3$. Then X_i are the only orthogonal pairs in $X = X_1 \cup X_2 \cup X_3$. Since $\sigma X = X$ we see that σ induces a permutation on the set $\{X_1, X_2, X_3\}$.

The above shows that $\mathbb{O}(\hat{L})$ induces Sym_3 on this 3-set. We may therefore assume that σ preserves each X_i. In this case σ acts trivially on $\alpha_1(-1)^2, \alpha_2(-1)^2, (\alpha_1 + \alpha_2)(-1)^2$. That is, σ acts trivially on the subVOA they generate, which is isomorphic to $M(1)^+$. As a result, σ is in the LVOA$^+$-group. $\square$

6.3 Alternate proof for $b = 2$

The system of equations in the variables c_i which occurred in the proof of 6.4 can be replaced by an equivalent system 6.7 which looks more symmetric. The old system was solved with software package MacCauley but not with Maple. The new system was solved with Maple and gives the same result as before.

Notation 6.6. Let r and s be independent norm 4 elements so that $t := -r - s$ has norm 4. Let w be an idempotent $w = p + q$, where $p = ar^2 + bs^2 + ct^2$ and $q = dv_r + ev_s + ev_t$ which satisfies $(w, w) = \frac{1}{16}$. Since $(L, L) \leq 2\mathbb{Z}$, we may and do assume that the epsilon-function is identically 1. It follows that $v_r \times v_s = v_t$ and similarly for all permutations of $\{r, s, t\}$.

Lemma 6.7. *From $w^2 = w$, we have equations*

$$(e1) \qquad a = 16a^2 + 4ab + 4ac - 4bc + d^2$$

$$(e2) \qquad b = 16b^2 + 4bc + 4ba - 4ac + e^2$$

$$(e3) \qquad c = 16c^2 + 4cq + 4cb - 4ab + f^2$$

$$(e4) \qquad d = 2d(16a + 4b + 4c) + 2ef$$

$$(e5) \qquad e = 2e(4a + 16b + 4c) + 2df$$

$$(e6) \qquad f = 2f(4a + 4b + 16c) + 2de$$

and from $(w, w) = \frac{1}{16}$*, we get the equation*

$$(e7) \qquad \frac{1}{16} = 32(a^2 + b^2 + c^2) + 16(ab + ac + bc) + 2(d^2 + e^2 + f^2).$$

Proof. Straightforward from Appendix: Algebraic rules. $\square$

Proposition 6.8. *There are just 6 solutions* $(a, b, c, d, e, f) \in \mathbb{C}^6$ *to the equations* $(e1), \ldots, (e7)$*. They are* $(\frac{1}{32}, 0, 0, \frac{1}{8}, 0, 00), (\frac{1}{32}, 0, 0, -\frac{1}{8}, 0, 00)$ *and ones obtained from these by powers of the permutation* $(abc)(def)$*.*

Proof. This follows from use of the `solve` command in the software package Maple. $\square$

Remark 6.9. If we omit (e7), there are infinitely many solutions with $d = e = f = 0$. The reason is that the Jordan algebra of symmetric degree 2 matrices has infinitely many idempotents. It seems possible that the system in Lemma 6.7 could be solved by hand.

6.4 $L_1 = \emptyset$ and $rank(L_2) = 2$; $b = 1$.

We now deal with the cases $b = 1$ in the Gram matrix.

Proposition 6.10. *If* $b = 1$ *in the Gram matrix, then* $Aut(V_L^+)$ *is the* $LVOA^+$ *group.*

Proof. By Corollary 5.27, any automorphism of V_L^+ preserves $M(1)_2^+$, the result follows from Lemma 3.3 $\square$

7 $Aut(V_L^+)$, for $L_1 = \emptyset$, $rank(L_2) = 1$

In this case we can assume that $L_2 = \{2\alpha_1, -2\alpha_1\}$. Let $\alpha_2 \in H$ such that $(\alpha_i, \alpha_j) = \delta_{i,j}$. Then $(V_L^+)_2$ is 4-dimensional with basis $v_{2\alpha_1}, \frac{1}{2}\alpha_1(-1)^2, \frac{1}{2}\alpha_2(-1)^2, \alpha_1(-1)\alpha_2(-1)$.

Lemma 7.1. *Any automorphism of V_L^+ preserves the subspace S^2H of $(V_L^+)_2$ spanned by $\frac{1}{2}\alpha_1(-1)^2, \frac{1}{2}\alpha_2(-1)^2, \alpha_1(-1)\alpha_2(-1)$.*

Proof. Since Virasoro vectors of central charge 1 in S^2H span S^2H, it is enough to show that any Virasoro vector of central charge 1 lies in S^2H.

Let $t = d_1\frac{\alpha_1^2}{2} + d_2\frac{\alpha_2^2}{2} + d_3 v_{2\alpha_1} + d_4\alpha_1\alpha_2$ be a Virasoro vector of central charge 1 with $d_3 \neq 0$. Then we must have $t \times t = 2t$ and $(t,t) = 1/2$. A straightforward computation shows that

$$t \times t = d_1^2\alpha_1^2 + d_2^2\alpha_2^2 + d_3^2(2\alpha_1)^2 + d_4^2(\alpha_1^2 + \alpha_2^2) + 4d_1d_3v_{2\alpha_1} + 2d_1d_4\alpha_1\alpha_2 + 2d_2d_4\alpha_1\alpha_2.$$

This gives four equations

$$\begin{aligned} d_1 &= d_1^2 + 4d_3^2 + d_4^2 \\ d_2 &= d_2^2 + d_4^2 \\ d_3 &= 2d_1d_3 \\ d_4 &= d_1d_4 + d_2d_4. \end{aligned}$$

The relation $(t,t) = 1/2$ gives one more equation:

$$\frac{1}{2} = \frac{1}{2}d_1^2 + \frac{1}{2}d_2^2 + d_4^2 + 2d_3^2.$$

Thus

$$1 = d_1 + d_2.$$

Since $d_3 \neq 0$, $d_1 = 1/2$ and $d_2 = 1/2$. So we have

$$\frac{1}{4} = 4d_3^2 + d_4^2, \frac{1}{4} = 2d_3^2 + d_4^2.$$

This forces $d_3 = 0$, a contradiction. □

Proposition 7.2. *In this case, $Aut(V_L^+)$ is the $LVOA^+$-group.*

Proof. By Corollary 5.27, any automorphism of V_L^+ preserves $M(1)_2^+$, the result follows from Lemma 3.3 □

8 $Aut(V_L^+)$, for $L_1 \neq \emptyset$

Finally we deal with the case that $L_1 \neq \emptyset$. There are two cases: $\mathrm{rank}(L_1) = 2$ or $\mathrm{rank}(L_1) = 1$.

8.1 $rank(L_1) = 2$

In this case $L = L_{A_1^2}$ or $L = L_{A_2}$ because these are the only rank 2 root lattices possible and each is a maximal even integral lattice in its rational span.

8.1.1 L has type A_1^2

If $L \cong L_{A_1^2}$ then $Aut(V_L) \cong PSL(2,\mathbb{C}) \wr 2$ and

$$V_L^+ \cong V_{L_{A_1}}^+ \otimes V_{L_{A_1}}^+ \oplus V_{L_{A_1}}^- \otimes V_{L_{A_1}}^-.$$

Since the connected component of the identity in $Aut(V_L)$ contains a lift of -1_L, we may assume that such a lift is in a given maximal torus, so is equal to the automorphism $e^{\pi i\beta(0)/2}$, where β is a sum of orthogonal roots.

It follows that $V_L^+ \cong V_K$, where $K = 2L + \mathbb{Z}\beta$. The result [DN1] implies that $Aut(V_L^+) \cong Aut(V_K)$, which is the LVOA group $\mathbb{T}_2.Dih_8$.

8.1.2 L has type A_2

Here, $(V_L^+)_1$ is a 3-dimensional Lie algebra isomorphic to $sl(2,\mathbb{C})$. The difficult part in this case is to determine the vertex operator subalgebra generated by $(V_L^+)_1$. Let $L_{A_2} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ such that $(\alpha_i,\alpha_i) = 2$ and $(\alpha_1,\alpha_2) = -1$. The set of roots in L is $L_1 = \{\pm\alpha_i | i = 1,2,3\}$ where $\alpha_3 = \alpha_1+\alpha_2$. The positive roots are $\{\alpha_i | i = 1,2,3\}$. The space $(V_L^+)_1$ is 3-dimensional with a basis v_{α_i} for $i = 1,2,3$ and $(V_L^-)_1$ is 5-dimensional with a basis $\alpha_1(-1), \alpha_2(-1), e^{\alpha_i} - e^{-\alpha_i}$ for $i = 1,2,3$. It is a straightforward to verify that $(v_{\alpha_i})_{-1}v_{\alpha_i}$ for $i = 1,2,3$ and $\alpha_i(-1)^2$ for $i = 1,2,3$ span the same space. Thus $\omega = \frac{1}{4}\alpha_1(-1)^2 + \frac{1}{12}(\alpha_1(-1) + 2\alpha_2(-1))^2$ lies in the vertex operator algebra generated by $(V_L^+)_1$.

In order to determine the vertex operator algebra generated by $(V_L^+)_1$ we need to recall the standard modules for affine algebra

$$A_1^{(1)} = \hat{sl}(2,\mathbb{C}) = sl(2,\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$$

(cf. [DL]). We use the standard basis $\{\alpha, x_\alpha, x_{-\alpha}\}$ for $sl(2,\mathbb{C})$ such that

$$[\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}, [x_\alpha, x_{-\alpha}] = \alpha.$$

We fix an invariant symmetric nondegenerate bilinear form on $sl(2,\mathbb{C})$ such that $(\alpha,\alpha)=2$. The level k standard $A_1^{(1)}$-modules are parametrized by dominant integral linear weights $\frac{i}{2}\alpha$ for $i=0,...,l$ such that the highest weight of the $A_1^{(1)}$-module, viewed as a linear form on $\mathbb{C}\alpha\oplus\mathbb{C}K\subset \hat{sl}(2,\mathbb{C})$, is given by $\frac{i}{2}\alpha$ and the correspondence $K\mapsto k$. Let us denote the corresponding standard $A_1^{(1)}$-module by $L(k,\frac{i}{2}\alpha)$. It is well known that $L(k,0)$ is a simple rational vertex operator algebra and $L(k,\frac{i}{2}\alpha)$ for $i=0,...,k$ is a complete list of irreducible $L(k,0)$-modules (cf. [DL], [FZ] and [L2]). Note that

$$L(k,\frac{i}{2}\alpha)=\oplus_{n=0}^{\infty}L(k,\frac{i}{2}\alpha)_{\lambda_i+n}$$

where $\lambda_i=\frac{i(i+2)}{4(k+2)}$ and $L(k,\frac{i}{2}\alpha)_{\lambda_i+n}$ is the eigenspace of $L(0)$ with eigenvalue λ_i+n (cf. [DL]). In fact, the lowest weight space $L(k,\frac{i}{2}\alpha)_{\lambda_i}$ of $L(k,\frac{i}{2}\alpha)$ is an irreducible $sl(2,\mathbb{C})$-module of dimension $i+1$.

Since $V_{L_{A_2}}$ is a unitary module for affine algebra $A_2^{(1)}$ (cf. [FK]), the vertex operator algebra V generated by $(V_L^+)_1$ is isomorphic to the standard level k $A_1^{(1)}$-module $L(k,0)$ for some nonnegative integer k. Let $\{v_1,v_2,v_3\}$ be an orthonormal basis of $sl(2,\mathbb{C})$ with respect to the standard bilinear form. Then $\omega'=\frac{1}{2(k+2)}\sum_{i=1}^3 v_i(-1)^2\mathbf{1}\in V$ is the Segal-Sugawara Virasoro vector. Let

$$Y(\omega',z)=\sum_{n\in\mathbb{Z}}L(n)'z^{-n-2}.$$

Then

$$[L(n)-L(n)',u_m]=0$$

for $m,n\in\mathbb{Z}$ and $u\in V$. So $L(-2)-L(-2)'$ acts as a constant on V as V is a simple vertex operator algebra. As a result, $L(-2)-L(-2)'=0$ since the left side is both a constant and an operator which shifts degree by 2. The creation axiom for VOAs implies that, $\omega'=\omega$. Since the central charge of ω is 2, the central charge $\frac{3k}{2(k+2)}$ of ω' is also 2. This implies that $k=4$ and $V\cong L(4,0)$. Now V_L^+ is a $L(4,0)$-module and the quotient module V_L^+/V has minimal weight (as inherited from V_L^+) greater than 1. On the other hand, the minimal weight of the irreducible $L(4,0)$-module $L(4,\frac{i}{2}\alpha)$ is $\frac{i(i+2)}{4(4+2)}$ which is less than 2 for $0\le i\le 4$. Since every irreducible is one of these, we conclude $V_L^+=V=L(4,0)$. Since V_L^- is an irreducible V_L^+-module with minimal weight 1, we immediately see that $V_L^-=L(4,2\alpha)$.

So we have proved the following:

Proposition 8.1. *If $rank(L_1)=2$ there are two cases.*

(1) If $L=L_{A_1^2}$ then V_L^+ is again a lattice vertex operator algebra V_K where K is generated by β_1,β_2 with $(\beta_i,\beta_i)=4$ and $(\beta_1,\beta_2)=0$. The automorphism group of V_L^+ is the LVOA$^+$ group which is isomorphic to the LVOA-group for lattice K.

(2) If $L = L_{A_2}$, then V_L^+ is isomorphic to the vertex operator algebra $L(4,0)$ and $Aut(V_L^+)$ is isomorphic to $PSL(2,\mathbb{C})$ which is the automorphism group of $sl(2,\mathbb{C})$.

8.2 $rank(L_1) = 1$

8.2.1 L rectangular.

We first assume that $L = \mathbb{Z}r + \mathbb{Z}s$ such that $(r,r) = 2$, $(s,s) \in 6 + 8\mathbb{Z}$ and $(r,s) = 0$. Then $V_L = V_{L_{A_1}} \otimes V_{\mathbb{Z}s}$ and

$$V_L^+ = V_{L_{A_1}}^+ \otimes V_{\mathbb{Z}s}^+ \oplus V_{L_{A_1}}^- \otimes V_{\mathbb{Z}s}^-.$$

Lemma 8.2. *A group of shape $(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta \cdot \mathbb{Z}_2) \times \mathbb{Z}_2$ acts on V_L^+ as automorphisms.*

Proof. We have already mentioned that $V_{L_{A_1}}^+$ is isomorphic to $V_{\mathbb{Z}\beta}$ for $(\beta,\beta) = 8$ and $V_{L_{A_1}}^-$ is isomorphic to $V_{\mathbb{Z}\beta+\frac{1}{2}\beta}$ as $V_{L_{A_1}}^+$-modules. We also know from [DN1] that $Aut(V_{\mathbb{Z}\beta})$ is isomorphic to $\mathbb{C}\beta/(\mathbb{Z}\frac{1}{8}\beta) \cdot \mathbb{Z}_2$ where the generator of $\mathbb{Z}_2$ is induced from the -1 isometry of the lattice $\mathbb{Z}\beta$. The action of $\lambda\beta \in \mathbb{C}\beta$ is given by the operator $e^{2\pi i\lambda\beta(0)}$. Note that $\mathbb{C}\beta$ acts on $V_{\mathbb{Z}\beta+\frac{1}{2}\beta}$ in the same way. But the kernel of the action of $\mathbb{C}\beta$ on $V_{\mathbb{Z}\beta+\frac{1}{2}\beta}$ is $\mathbb{Z}\frac{1}{4}\beta$ instead of $\mathbb{Z}\frac{1}{8}\beta$. As a result, the torus $\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta$ acts on both $V_{\mathbb{Z}\beta}$ and $V_{\mathbb{Z}\beta+\frac{1}{2}\beta}$. By [DG], $Aut(V_{\mathbb{Z}s}^+)$ is isomorphic to $\frac{1}{2}\mathbb{Z}s/\mathbb{Z}s \cong \mathbb{Z}_2$ which also acts on $V_{\mathbb{Z}s}^-$. So the group $(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta \cdot \mathbb{Z}_2) \times \mathbb{Z}_2$ acts on V_L^+ as automorphisms. $\square$

In order to determine $Aut(V_L^+)$ in this case we need to recall the notion of commutant from [FZ].

Definition 8.3. Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $U = (U, Y, \mathbf{1}, \omega')$ be vertex operator subalgebra with a different Virasoro vector ω'. The *commutant* U^c of U in V is defined by

$$U^c := \{v \in V | u_n v = 0, u \in U, n \geq 0\}.$$

Remark 8.4. The above space U^c is the space of vacuum-like vectors for U (see [L1]).

Lemma 8.5. *Let V be a vertex operator algebra and $U^i = (U^i, Y, 1, \omega^i)$ are simple vertex operator subalgebras of V with Virasoro vector ω^i for $i = 1, 2$ such that $\omega = \omega^1 + \omega^2$. We assume that V has a decomposition*

$$V \cong \oplus_{i=0}^p P^i \otimes Q^i$$

as $U^1 \otimes U^2$-module such that $P^0 \cong U^1$, $Q^0 \cong U^2$, the P^i are inequivalent U^1-modules and the Q^i are inequivalent U^2-modules. Then $(U^1)^c = U^2$ and $(U^2)^c = U^1$.

Proof. It is enough to prove that $(U^2)^c \subset U^1$. Let $v \in (U^2)^c$. Then v is a vacuum-like vector for U^2. Then the U^2-submodule generated by v is isomorphic to U^2 (see [L1]). Since V is a completely reducible U^2-module and any U^2-submodule isomorphic to U^2 is contained in $U^1 \otimes U^2$. In particular, $v \in U^1 \otimes U^2$. This forces $v \in U^1$. $\square$

Proposition 8.6. *The group $Aut(V_L^+)$ is isomorphic to $((\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta) \cdot \mathbb{Z}_2) \times \mathbb{Z}_2$. This can be interpreted as an action of $\mathbb{N}(\widehat{\mathbb{Z}\beta}) \times \mathbb{Z}_2$, where $(\beta, \beta) = 8$.*

Proof. We have already shown 8.2 that the group $((\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta) \cdot \mathbb{Z}_2) \times \mathbb{Z}_2$ acts on V_L^+ as automorphisms.

Let σ be an automorphism of V_L^+. Then $\sigma\beta(-1) = \lambda\beta(-1)$ for some nonzero $\lambda \in \mathbb{C}$ as $(V_L^+)_1$ is spanned by $\beta(-1)$. This implies that $\sigma\beta(n)\sigma^{-1} = \lambda\beta(n)$ for $n \in \mathbb{Z}$. Since $V_{\mathbb{Z}s}^+$ is precisely the subspace of V_L^+ consisting of vectors killed by $\beta(n)$ for $n \geq 0$, we see that $\sigma V_{\mathbb{Z}s}^+ \subset V_{\mathbb{Z}s}^+$. Thus $\sigma|_{V_{\mathbb{Z}s}^+}$ is an automorphism of $V_{\mathbb{Z}s}^+$. On the other hand, $V_{L_{A_1}}^+$ is the commutant of $V_{\mathbb{Z}s}^+$ in $V_L^+$by Lemma 8.5.

The above show that σ induces an automorphism of the tensor factor $V_{L_{A_1}}^+$. The restriction of σ to $V_{L_{A_1}}^+ \otimes V_{\mathbb{Z}s}^+$ is a product $\sigma_1 \otimes \sigma_2$ for some $\sigma_1 \in Aut(V_{L_{A_1}}^+)$ and $\sigma_2 \in Aut(V_{\mathbb{Z}s}^+)$. Multiplying σ by σ_2 we can assume that $\sigma = 1$ on $V_{\mathbb{Z}s}^+$. As we have already mentioned, $Aut(V_{L_{A_1}}^+)$ is isomorphic to $(\mathbb{C}\beta/\mathbb{Z}\frac{1}{8}\beta)\cdot\mathbb{Z}_2$. Since $(\mathbb{C}\beta/\mathbb{Z}\frac{1}{8}\beta)$ acts trivially on $\beta(-1)$ and the outer factor $\mathbb{Z}_2$ is represented in $Aut(V_L^+)$ by action of ± 1 on $\beta(-1)$. As a result $\sigma\beta(-1) = \pm\beta(-1)$. Now multiplying σ by an outer element of $(\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta) \cdot \mathbb{Z}_2$, we can assume that $\sigma\beta(-1) = \beta(-1)$.

Set $W_{n\beta} = M(1) \otimes e^{n\beta} \otimes V_{\mathbb{Z}s}^+$ and $W_{n\beta+\beta/2} = M(1) \otimes e^{n\beta+\beta/2} \otimes V_{\mathbb{Z}s}^-$ for $n \in \mathbb{Z}$ where $M(1) = \mathbb{C}[\beta(-n)|n > 0]$. Then $V_L^+ = \oplus_{n\in\frac{1}{2}\mathbb{Z}} W_{n\beta}$ and $u_m v \in W_{\mu+\nu}$ for $u \in W_\mu$ and $v \in W_\nu$, and $n \in \mathbb{Z}$. Note that W_μ is the eigenspace of $\beta(0)$ with eigenvalue (β, μ). Since $\sigma\beta(-1) = \beta(-1)$ we see that σ acts on each W_μ as a constant λ_μ and $\lambda_\mu\lambda_\nu = \lambda_{\mu+\nu}$. As a result, $\sigma = e^{2\pi i\gamma(0)}$ for some $\gamma \in \mathbb{C}\beta$. That is, σ lies in $\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta$. This completes the proof. $\square$

8.2.2 L not rectangular

Next we assume that $L \neq \mathbb{Z}r \perp \mathbb{Z}s$. Then $L = \mathbb{Z}r \oplus \mathbb{Z}\frac{1}{2}(s+t)$ where $(s, s) \in 6 + 8\mathbb{Z}$ and $(s, s) \geq 14$ (see 4.3). Let $K = \mathbb{Z}r \oplus \mathbb{Z}s$. Then $L = K \cup (K + \frac{1}{2}(r+s))$ and $V_L = V_{\mathbb{Z}r} \otimes V_{\mathbb{Z}s} \oplus V_{(\mathbb{Z}+\frac{1}{2})r} \otimes V_{(\mathbb{Z}+\frac{1}{2})s}$. Thus

$$V_L^+ = V_{\mathbb{Z}r}^+ \otimes V_{\mathbb{Z}s}^+ \oplus V_{\mathbb{Z}r}^- \otimes V_{\mathbb{Z}s}^- \oplus V_{(\mathbb{Z}+\frac{1}{2})r}^+ \otimes V_{(\mathbb{Z}+\frac{1}{2})s}^+ \oplus V_{(\mathbb{Z}+\frac{1}{2})r}^- \otimes V_{(\mathbb{Z}+\frac{1}{2})s}^-$$

and

$$V_L^+ = V_K^+ \oplus V_{K+\frac{1}{2}(s+t)}^+.$$

As before, we note that $V_{\mathbb{Z}r}^{+}$ is isomorphic to $V_{\mathbb{Z}\beta}$ with $(\beta, \beta) = 8$.

Proposition 8.7. *Assume that* $rank(L_1) = 1$, $L \neq \mathbb{Z}r + \mathbb{Z}s$, r, s *as above. Then* $Aut(V_L^+) \cong (\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta) \cdot \mathbb{Z}_2$, *where* $(\beta, \beta) = 8$. *The action is trivial on the subVOA* $V_{\mathbb{Z}s}^{+}$ *and leaves* $V_{\mathbb{Z}r}^{+}$ *invariant. A generator of the quotient* $\mathbb{Z}_2$ *comes from the* -1 *isometry of* $\frac{1}{4}\mathbb{Z}\beta$ *and* $\alpha \in \mathbb{C}\beta$ *acts as* $e^{2\pi i\alpha(0)}$.

Proof. Note that V_K^+ is a subalgebra of V_L^+ and $V_{K+\frac{1}{2}(r+s)}^{+}$ is an irreducible V_K^+-module. By Proposition 8.6,

$$Aut(V_K^+) = ((\mathbb{C}\beta/\frac{1}{4}\mathbb{Z}\beta)\cdot\mathbb{Z}_2) \times \mathbb{Z}_2.$$

As we have already mentioned that $V_{\mathbb{Z}r}^{+}$ is isomorphic to $V_{\mathbb{Z}\beta}$ with $(\beta, \beta) = 8$ and $V_{\mathbb{Z}r}^{-}$ is isomorphic to $V_{\mathbb{Z}\beta+\frac{1}{2}\beta}$ as $V_{\mathbb{Z}\beta}$-module. It is easy to see that $V_{(\mathbb{Z}+\frac{1}{2})\beta}^{\pm}$ is isomorphic to $V_{(\mathbb{Z}\pm\frac{1}{4})\beta}$ as $V_{\mathbb{Z}\beta}$-module. So the action of $\mathbb{C}\beta/\mathbb{Z}\frac{1}{4}\beta$ on V_K^+ cannot be extended to an action of V_L^+. But the torus $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta$ does acts on V_L^+. As a result, $\mathbb{N}(\widehat{\mathbb{Z}\frac{1}{2}\beta}) \cong (\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta) \cdot \mathbb{Z}_2$ is a subgroup of $Aut(V_L^+)$.

The same argument used in the proof of Proposition 8.6 shows that that any automorphism σ of V_L^+ preserves $V_{\mathbb{Z}r}^{+} \otimes V_{\mathbb{Z}s}^{+}$. Since $V_{\mathbb{Z}r}^{+} \otimes V_{\mathbb{Z}s}^{+}$, $V_{\mathbb{Z}r}^{-} \otimes V_{\mathbb{Z}s}^{-}$, $V_{(\mathbb{Z}+\frac{1}{2})r}^{+} \otimes V_{(\mathbb{Z}+\frac{1}{2})s}^{+}$, $V_{(\mathbb{Z}+\frac{1}{2})r}^{-} \otimes V_{(\mathbb{Z}+\frac{1}{2})s}^{-}$ are inequivalent irreducible $V_{\mathbb{Z}r}^{+} \otimes V_{\mathbb{Z}s}^{+}$-modules (see [DM1] and [DLM]), we see that σ preserves

$$V_K^+ = V_{\mathbb{Z}r}^{+} \otimes V_{\mathbb{Z}s}^{+} \oplus V_{\mathbb{Z}r}^{-} \otimes V_{\mathbb{Z}s}^{-}.$$

Since $\mathbb{C}\beta/\frac{1}{4}\mathbb{Z}\beta \cdot \mathbb{Z}_2$ is a quotient group of $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta \cdot \mathbb{Z}_2$, we can multiply σ by an element of $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta \cdot \mathbb{Z}_2$ and assume that σ acts trivially on the first tensor factor of V_K^+. If σ is the identity on V_K^+, then σ is either 1 or -1 on $V_{K+\frac{1}{2}(r+s)}^{+}$. If σ is -1 on $V_{K+\frac{1}{2}(r+s)}^{+}$ then $\sigma = e^{\pi i\frac{1}{2}\beta(0)}$ is an element of $\mathbb{C}\beta/\frac{1}{2}\mathbb{Z}\beta \cdot \mathbb{Z}_2$.

If σ is not identity on V_K^+ then we must have $\sigma = e^{\pi i\frac{1}{(s,s)}s(0)}$ on V_K^+. We will get a contradiction in this case. Notice that the lowest weight space of $V_{(\mathbb{Z}+\frac{1}{2})r}^{+} \otimes V_{(\mathbb{Z}+\frac{1}{2})s}^{+}$ is 1-dimensional and spanned by $u = (e^{r/2} + e^{-r/2}) \otimes (e^{s/2} + e^{-s/2})$. Since σ preserves $V_{(\mathbb{Z}+\frac{1}{2})r}^{+} \otimes V_{(\mathbb{Z}+\frac{1}{2})s}^{+}$, it must map u to λu for some nonzero constant λ. Note that $u_{\frac{1}{4}(r+s,r+s)-1}u = 4$. This forces $\lambda = \pm 1$. On the other hand,

$$u_{-\frac{1}{4}(r+s,r+s)-1}u = (e^r + e^{-r}) \otimes (e^s + e^{-s}) + \cdots$$

has nontrivial projection to the -1 eigenspace of σ in V_K^+. This forces $\lambda = \pm i$, a contradiction. □

9 Appendix: Algebraic rules

For the symmetric matrices of degree n, there is a widely used basis, Jordan product and inner product, which we review here. (This section is taken almost verbatim from [G4]).

Proposition 9.1. *H is a vector space of finite dimension n with nondegenerate symmetric bilinear form $(\cdot,\cdot)$.*

$r, s, \dots$ stand for elements of H and rs stands for the symmetric tensor $r\otimes s+s\otimes r$.

$rs \times pq = (r,p)sq + (r,q)sp + (s,p)rq + (s,q)rp$.

$(rs, pq) = (r,p)(s,q) + (r,q)(s,p)$

$rs \times v_t = (r,t)(s,t)v_t$.

Definition 9.2. *The Symmetric Bilinear Form.* Source: [FLM], p.217. This form is associative with respect to the product (Section 3). We write H for H_1. The set of all g^2 and x_α^+ spans V_2.

(2.2.1) $$\langle g^2, h^2\rangle = 2\langle g, h\rangle^2,$$

whence

(2.2.2) $$\langle pq, rs\rangle = \langle p, r\rangle\langle q, s\rangle + \langle p, s\rangle\langle q, r\rangle, \text{ for } p, q, r, s \in H.$$

(2.2.3) $$\langle x_\alpha^+, x_\beta^+\rangle = \begin{cases} 2 & \alpha = \pm\beta \\ 0 & else \end{cases}$$

(2.2.4) $$\langle g^2, x_\beta^+\rangle = 0.$$

Definition 9.3. In addition, we have the distinguished Virasoro element ω and identity $\mathbb{I} := \frac{1}{2}\omega$ on V_2 (see Section 3). If h_i is a basis for H and h_i^* the dual basis, then $\omega = \frac{1}{2}\sum_i h_i h_i^*$.

Remark 9.4.

(2.4.1) $$\langle g^2, \omega\rangle = \langle g, g\rangle$$

(2.4.2) $$\langle g^2, \mathbb{I}\rangle = \frac{1}{2}\langle g, g\rangle$$

(2.4.3)
$$\langle \mathbb{I}, \mathbb{I} \rangle = dim(H)/8$$

(2.4.4)
$$\langle \omega, \omega \rangle = dim(H)/2$$

If $\{x_i \mid i = 1, \ldots \ell\}$ is an ON basis,

(2.4.5)
$$\mathbb{I} = \frac{1}{4} \sum_{i=0}^{\ell} x_i^2$$

(2.4.6)
$$\omega = \frac{1}{2} \sum_{i=0}^{\ell} x_i^2.$$

Definition 9.5. *The product on* V_2^F comes from the vertex operations. We give it on standard basis vectors, namely $xy \in S^2H_1$, for $x, y \in H_1$ and $v_\lambda := e^\lambda + e^{-\lambda}$, for $\lambda \in L_2$. (This is the same as x_λ^+, used in [FLM].) Note that (3.1.1) give the Jordan algebra structure on S^2H_1, identified with the space of symmetric 8×8 matrices, and with $\langle x, y \rangle = \frac{1}{8} tr(xy)$. The function ε below is a standard part of notation for lattice VOAs.

(3.1.1)
$$x^2 \times y^2 = 4\langle x, y \rangle xy, \qquad pq \times y^2 = 2\langle p, y \rangle qy + 2\langle q, y \rangle py,$$
$$pq \times rs = \langle p, r \rangle qs + \langle p, s \rangle qr + \langle q, r \rangle ps + \langle q, s \rangle pr;$$

(3.1.2)
$$x^2 \times v_\lambda = \langle x, \lambda \rangle^2 v_\lambda, \qquad xy \times v_\lambda = \langle x, \lambda \rangle \langle y, \lambda \rangle v_\lambda$$

(3.1.3)
$$v_\lambda \times v_\mu = \begin{cases} 0 & \langle \lambda, \mu \rangle \in \{0, \pm 1, \pm 3\}; \\ \varepsilon \langle \lambda, \mu \rangle v_{\lambda+\mu} & \langle \lambda, \mu \rangle = -2; \\ \lambda^2 & \lambda = \mu. \end{cases}$$

Some consequences are these:

Corollary 9.6. *If $x_1, \ldots$ is a basis and $y_1, \ldots$ is the dual basis, then $\mathbb{I} := \frac{1}{4} \sum_{i=1}^n x_i y_i$ is the identity of the algebra S^2H.*

$(\mathbb{I}, \mathbb{I}) = \frac{n}{8}$.

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